Probabilistic Dense Coding Using Non-Maximally Entangled Three-Particle State

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We present a scheme of probabilistic dense coding via a quantum channel of non-maximally entangled three-particle state. The quantum dense coding will be succeeded with a certain probability if the sender introduces an auxiliary particle and performs a collective unitary transformation. Furthermore, the average information transmitted in this scheme is calculated.

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One of the essential features of quantum information is its capacity for entanglement. The present-day entanglement theory has its roots in the key discoveries: quantum teleportation,[1] quantum cryptography with Bell theorem,[2] and quantum dense coding.[3] Entanglement has also played important role in development of quantum computation and quantum communication. [4−7]

Holevo has shown that one qubit can carry at most only one bit of classical information. [8] In 1992, Bennett and Wiesner discovered a fundamental primitive, quantum dense coding, [3] which allows to communicate two classical bits by sending one a priori entangled qubit. Quantum dense coding is one of many surprising applications of quantum entanglement. In 1996, quantum dense coding was experimentally demonstrated with polarization entangled photons for the case of discrete variables by Mattle et al. [9] Recent years, some schemes for quantum dense coding using multi-particle entangled states via local measurements,[10] GHZ state,[11] nonsymmetric quantum channel [12] were proposed. Liu et al.[13] presented the general scheme for dense coding between multi-parties using a high-dimensional state. All these cases deal with maximally entangled states.

Recently, Hao et al.[14] gave a probabilistic dense coding scheme using the two-qubit pure state $|\phi\rangle = a|00\rangle + b|11\rangle$. A general probabilistic dense coding scheme was put forward by Wang et al [15].

In this Letter, we suggest a scheme of probabilistic dense coding via a quantum channel of non-maximally entangled three-particle state. The average information transmitted in the scheme is calculated. Furthermore, the scheme is generalized to d-level ($d > 3$) for three parties.

We first consider the dense coding between three-parties (Alice, Bob and Charlie) via a maximally entangled three-particle state. Suppose Alice and Bob are the senders, Charlie is the receiver, a maximally entangled three-particle state

$$|\psi_{00}\rangle_{ABC} = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)_{ABC} \qquad (1)$$

is shared by them, and the three particles A, B and C are held by Alice, Bob and Charlie, respectively.

Let us introduce the nine single-particle operations as follows:

$$U_{00} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_{01} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \end{pmatrix},$$
$$U_{02} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{4\pi i/3} & 0 \\ 0 & 0 & e^{2\pi i/3} \end{pmatrix}, \quad U_{10} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$U_{11} = \begin{pmatrix} 0 & 0 & e^{4\pi i/3} \\ 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \end{pmatrix}, \quad U_{12} = \begin{pmatrix} 0 & 0 & e^{2\pi i/3} \\ 1 & 0 & 0 \\ 0 & e^{4\pi i/3} & 0 \end{pmatrix},$$
$$U_{20} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad U_{21} = \begin{pmatrix} 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \\ 1 & 0 & 0 \end{pmatrix},$$
$$U_{22} = \begin{pmatrix} 0 & e^{4\pi i/3} & 0 \\ 0 & 0 & e^{2\pi i/3} \\ 1 & 0 & 0 \end{pmatrix}. \qquad (2)$$

It is easy to prove that

$$\begin{aligned}
& U_{00}(A)\otimes U_{00}(B)|\psi_{00}\rangle_{ABC} \qquad (3)\\
= \; & \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)_{ABC} \equiv |\psi^0_{00}\rangle_{ABC},\\
& U_{00}(A)\otimes U_{10}(B)|\psi_{00}\rangle_{ABC}\\
= \; & \frac{1}{\sqrt{3}}(|010\rangle + |121\rangle + |202\rangle)_{ABC} \equiv |\psi^0_{01}\rangle_{ABC},\\
& U_{00}(A)\otimes U_{20}(B)|\psi_{00}\rangle_{ABC}\\
= \; & \frac{1}{\sqrt{3}}(|020\rangle + |101\rangle + |212\rangle)_{ABC} \equiv |\psi^0_{02}\rangle_{ABC},\\
& U_{01}(A)\otimes U_{00}(B)|\psi_{00}\rangle_{ABC}\\
= \; & \frac{1}{\sqrt{3}}(|000\rangle + e^{2\pi i/3}|111\rangle + e^{4\pi i/3}|222\rangle)_{ABC} \equiv |\psi^0_{10}\rangle_{ABC},\\
& U_{01}(A)\otimes U_{10}(B)|\psi_{00}\rangle_{ABC}\\
= \; & \frac{1}{\sqrt{3}}(|010\rangle + e^{2\pi i/3}|121\rangle + e^{4\pi i/3}|202\rangle)_{ABC} \equiv |\psi^0_{11}\rangle_{ABC},\\
& \cdots,\\
& U_{22}(A)\otimes U_{20}(B)|\psi_{00}\rangle_{ABC}\\
= \; & \frac{1}{\sqrt{3}}(|220\rangle + e^{4\pi i/3}|001\rangle + e^{2\pi i/3}|112\rangle)_{ABC} \equiv |\psi^2_{22}\rangle_{ABC}
\end{aligned}$$

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and $\{|\psi^0_{00}\rangle, |\psi^0_{01}\rangle, |\psi^0_{02}\rangle, |\psi^0_{10}\rangle, |\psi^0_{11}\rangle, \cdots, |\psi^2_{22}\rangle\}$ is a basis of the Hilbert space of the particles A, B and C.

Alice performs one of the nine unitary transformations stated in Eq.(2) on particle A, Bob operates one of three unitary transformations U_{00}, U_{10}, U_{20}, on particle B. Then they send their particles A and B to the receiver Charlie. After receiving the particles A and B, Charlie takes only one measurement in the basis $\{|\psi^0_{00}\rangle, |\psi^0_{01}\rangle, |\psi^0_{02}\rangle, |\psi^0_{10}\rangle, |\psi^0_{11}\rangle, \cdots, |\psi^2_{22}\rangle\}$, and she will know what operation Alice and Bob have carried out, that is, what messages are that Alice and Bob have encoded in the quantum state. Then Charlie can obtain $\log_2 27$ bits of information through only one measurement. So the dense coding is realized successfully.

In the following we will discuss a scheme of probabilistic dense coding between three parties via a non-maximally entangled three-particle state. We suppose that Alice, Bob and Charlie share a non-maximally entangled three-particle state

$$|\varphi\rangle_{ABC} = (x_0|000\rangle + x_1|111\rangle + x_2|222\rangle)_{ABC}, \tag{4}$$

where x_0, x_1, x_2 are real numbers, and $|x_0|^2 + |x_1|^2 + |x_2|^2 = 1$. Without loss of generality, we can suppose that $|x_0| \leq |x_1| \leq |x_2|$.

The scheme of probabilistic dense coding is composed of four steps.

Firstly, Alice introduces an auxiliary three-level particle a in the quantum state $|0\rangle_a$. Then she performs a unitary transformation U_{sim} on her particle A and auxiliary particle a under the basis $\{|00\rangle_{Aa}, |01\rangle_{Aa}, |02\rangle_{Aa}, |10\rangle_{Aa}, |11\rangle_{Aa}, |12\rangle_{Aa}, |20\rangle_{Aa}, |21\rangle_{Aa}, |22\rangle_{Aa}\}$:

$$U_{sim} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{01} & m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -m_{01} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{02} & 0 & M & -N \\ 0 & 0 & 0 & 0 & 0 & 0 & m_{02} & N & M \\ 0 & 0 & 0 & 0 & 0 & M & N & -m_{02} & 0 \\ 0 & 0 & 0 & 0 & 0 & -N & M & 0 & -m_{02} \end{pmatrix}, \tag{5}$$

where $M = \sqrt{(x_2^2 - x_1^2)/x_2^2}, N = \sqrt{(x_1^2 - x_0^2)/x_2^2}, m = \sqrt{1 - x_0^2/x_1^2}, m_{01} = x_0/x_1, m_{02} = x_0/x_2$. The collective unitary operator $U_{sim} \otimes I_{BC}$ (where I_{BC} is a 9×9 identity matrix) transforms the state $|\varphi\rangle_{ABC} \otimes |0\rangle_a$ into

$$\begin{aligned} & |\varphi'\rangle_{ABCa} \\ = \; & U_{sim} \otimes I_{BC}|\varphi\rangle_{ABC} \otimes |0\rangle_a \\ = \; & \sqrt{3}x_0 \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)_{ABC}|0\rangle_a \\ & + \sqrt{2}\sqrt{x_1^2 - x_0^2}\frac{1}{\sqrt{2}}(|111\rangle + |222\rangle)_{ABC}|1\rangle_a \\ & + \sqrt{x_2^2 - x_1^2}|222\rangle_{ABC}|2\rangle_a \\ \equiv \; & \sqrt{3}x_0|\varphi_{00}\rangle_{ABC}|0\rangle_a + \sqrt{2}\sqrt{x_1^2 - x_0^2}|\varphi'_{00}\rangle_{ABC}|1\rangle_a \\ & + \sqrt{x_2^2 - x_1^2}|\varphi''_{00}\rangle_{ABC}|2\rangle_a. \end{aligned} \tag{6}$$

Then Alice makes a measurement on the auxiliary particle a and tells Bob and Charlie her measurement result via a classical channel. If she gets the result $|0\rangle_a$, she ensures that the three particles A, B and C are in the maximally entangled three-particle state $|\varphi_{00}\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)$, the probability of obtaining $|0\rangle_a$ is $3x_0^2$ according to Eq.(6); if the result $|1\rangle_a$ is obtained, she ensures that the three particles A, B and C are in the state $|\varphi'_{00}\rangle = \frac{1}{\sqrt{2}}(|111\rangle + |222\rangle)$, and the probability of getting this result is $2(x_1^2 - x_0^2)$; if she gets the result $|2\rangle_a$, she knows that the three particles are in the product state $|\varphi''_{00}\rangle = |222\rangle$, and the probability of obtaining this result is $x_2^2 - x_1^2$.

Secondly, Alice and Bob encode classical information by performing the unitary transformations on their particle A and B respectively.

If the three particles A, B and C are in the state $|\varphi_{00}\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)$, one uses the dense coding protocol stated above. Here we do not recite any more.

If the three particles A, B and C are in the state $|\varphi'_{00}\rangle = \frac{1}{\sqrt{2}}(|111\rangle + |222\rangle)$, Alice encodes her message by performing one of six single-particle operations

$$\begin{aligned} U'_{00} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; U'_{01} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ U'_{10} &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; U'_{11} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ U'_{20} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \; U'_{21} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \end{aligned} \tag{7}$$

on particle A. However, Bob encodes his message only by operating one of three single-particle operations U'_{00}, U'_{10}, U'_{20} on particle B. Through simple calculation, we

can prove that

$$\begin{aligned} & U'_{00}(A) \otimes U'_{00}(B)|\varphi'_{00}\rangle_{ABC} \qquad (8) \\ = & \frac{1}{\sqrt{2}}(|111\rangle + |222\rangle)_{ABC} \equiv |\varphi'^{0}_{00}\rangle_{ABC}, \\ & U'_{00}(A) \otimes U'_{10}(B)|\varphi'_{00}\rangle_{ABC} \\ = & \frac{1}{\sqrt{2}}(|121\rangle + |202\rangle)_{ABC} \equiv |\varphi'^{1}_{00}\rangle_{ABC}, \\ & U'_{00}(A) \otimes U'_{20}(B)|\varphi'_{00}\rangle_{ABC} \\ = & \frac{1}{\sqrt{2}}(|101\rangle + |212\rangle)_{ABC} \equiv |\varphi'^{2}_{00}\rangle_{ABC}, \\ & U'_{01}(A) \otimes U'_{00}(B)|\varphi'_{00}\rangle_{ABC} \\ = & \frac{1}{\sqrt{2}}(|111\rangle - |222\rangle)_{ABC} \equiv |\varphi'^{0}_{01}\rangle_{ABC}, \\ & \cdots, \\ & U'_{21}(A) \otimes U'_{20}(B)|\varphi'_{00}\rangle_{ABC} \\ = & \frac{1}{\sqrt{2}}(|001\rangle - |112\rangle)_{ABC} \equiv |\varphi'^{2}_{21}\rangle_{ABC}. \end{aligned}$$

It is easy to see that the states in the set $\{|\varphi'^{k}_{mn}\rangle, m,k = 0,1,2; n = 0,1\}$ are orthogonal each other.

If the three particles A, B and C are in the product state $|\varphi''_{00}\rangle = |222\rangle$, Alice and Bob can encode their classical information by performing one of three single-particle operations on particles A and B independently:

$$U''_{00} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; U''_{10} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; U''_{20} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \qquad (9)$$

The state $|\varphi''_{00}\rangle$ will be transformed into the corresponding state respectively:

$$\begin{aligned} & \qquad (10) \\ & U''_{00}(A) \otimes U''_{00}(B)|\varphi''_{00}\rangle_{ABC} = |222\rangle_{ABC} \equiv |\varphi''_{00}\rangle_{ABC}, \\ & U''_{00}(A) \otimes U''_{10}(B)|\varphi''_{00}\rangle_{ABC} = |202\rangle_{ABC} \equiv |\varphi''_{01}\rangle_{ABC}, \\ & U''_{00}(A) \otimes U''_{20}(B)|\varphi''_{00}\rangle_{ABC} = |212\rangle_{ABC} \equiv |\varphi''_{02}\rangle_{ABC}, \\ & U''_{10}(A) \otimes U''_{00}(B)|\varphi''_{00}\rangle_{ABC} = |022\rangle_{ABC} \equiv |\varphi''_{10}\rangle_{ABC}, \\ & \cdots, \\ & U''_{20}(A) \otimes U''_{20}(B)|\varphi''_{00}\rangle_{ABC} = |112\rangle_{ABC} \equiv |\varphi''_{22}\rangle_{ABC}. \end{aligned}$$

Evidently, the states in the set $\{|\varphi''_{mn}\rangle, m,n = 0,1,2\}$ are orthogonal each other.

Thirdly, Alice and Bob send their particles A and B independently to Charlie.

Finally, After Charlie receives particle A and B, she takes only one measurement on the three particle A, B and C. The measurement basis is determined by Alice's measurement result. According to Charlie's measurement result, Charlie will know what operators Alice and Bob have carried out, i.e. he can obtain the classical information that Alice and Bob have encoded.

Apparently, the average information transmitted in this procedure is

$$I_{aver} = 3x_0^2 \log_2 27 + 2(x_1^2 - x_0^2)\log_2 18 + (x_2^2 - x_1^2)\log_2 9. \qquad (11)$$

In fact, the above protocol needs $2\log_2 3$ bits of classical information for Alice to tell Bob and Charlie her measurement result on the auxiliary particle. Obviously, when $x_0 = x_1 = x_2 = \frac{1}{\sqrt{3}}$, the three particles A, B and C is in the maximally entangled three-particle state, and the success probability of dense coding is one. The average information transmitted is $\log_2 27$ bits.

Now we would like to generalize the above protocol to d-level for three parties. Suppose that Alice, Bob and Charlie share a non-maximally entangled three-particle state

$$|\varphi\rangle_{ABC} = (x_0|000\rangle + x_1|111\rangle + \cdots + x_{d-1}|d-1d-1d-1\rangle)_{ABC}, \qquad (12)$$

where $x_0, x_1, \cdots, x_{d-1}$ are real numbers and satisfy $|x_0| \leq |x_1| \leq \cdots \leq |x_{d-1}|$.

The scheme of the probabilistic dense coding can be accomplished by four steps.

(1) Alice introduces an auxiliary d-level particle in the quantum state $|0\rangle_a$. Then she performs a proper unitary transformation on her particle A and the auxiliary particle. The collective unitary transformation $U_{sim} \otimes I_{BC}$ (where I_{BC} is a $d^2 \times d^2$ identity matrix) transforms the state $|\varphi\rangle_{ABC} \otimes |0\rangle_a$ into the state

$$\begin{aligned} & |\varphi\rangle_{ABCa} \qquad (13) \\ = & x_0(|000\rangle + |111\rangle + \cdots + |d-1d-1d-1\rangle)_{ABC}|0\rangle_a \\ & + \sqrt{x_1^2 - x_0^2}(|111\rangle + \cdots + |d-1d-1d-1\rangle)_{ABC}|1\rangle_a \\ & + \cdots + \sqrt{x_{d-1}^2 - x_{d-2}^2}|d-1d-1d-1\rangle_{ABC}|d-1\rangle_a. \end{aligned}$$

After that Alice performs a measurement on the auxiliary particle. The resulting state of the particles A, B and C will be respectively

$\frac{1}{\sqrt{d}}(|000\rangle + |111\rangle + \cdots + |d-1d-1d-1\rangle)$,
$\frac{1}{\sqrt{d-1}}(|111\rangle + \cdots + |d-1d-1d-1\rangle)$,
$\cdots$,
$|d-1d-1d-1\rangle$.

The probability of obtaining each resulting state is dx_0^2, $(d-1)(x_1^2 - x_0^2), \cdots, (x_{d-1}^2 - x_{d-2}^2)$, respectively.

(2) Alice tells Bob and Charlie her measurement result, then Alice and Bob encode classical information by making a unitary transformation on particle A and B respectively.

(3) Alice and Bob send particle A and B to Charlie respectively.

(4) After Charlie receives the particles A and B, she takes a measurement in the basis determined by Alice's measurement result. According to his measurement result, Charlie can obtain the classical information that Alice and Bob have encoded. The average information

transmitted is

$$
\begin{aligned}
& I_{aver} \\
= \ & dx_0^2\log_2 d^3 + (d-1)(x_1^2 - x_0^2)\log_2 d^2(d-1) \\
& +(d-2)(x_2^2 - x_1^2)\log_2 d^2(d-2) + \cdots \\
& +(x_{d-1}^2 - x_{d-2}^2)\log_2 d^2.
\end{aligned} \tag{14}
$$

Obviously, this probabilistic dense coding scheme needs $2\log_2 d$ bits of information to transmit Alice's measurement results on the auxiliary particle to Bob and Charlie.

It is easy to see that if the non-zero coefficients in Eq.(12) are totally equal, Alice does not need to introduce the auxiliary particle and makes unitary transformation U_{sim}. The classical information can be encoded directly by performing single-particle operations on particle A and B respectively. In this case the quantum state is called a deterministic quantum channel; otherwise, a probabilistic one if the coefficients are not equal totally. Obviously, a non-maximally entangled three-particle state is not equivalent to the probabilistic quantum channel. For example, in the $3 \otimes 3 \otimes 3$-dimensional case, the quantum state $\frac{1}{\sqrt{2}}(|111\rangle + |222\rangle)$ is not a maximally entangled three-particle state, but it is a deterministic quantum channel. So the first step in our protocol is to extract a series of deterministic quantum channels from a probabilistic one.

In summary, we have presented a scheme of probabilistic dense coding via a quantum channel of non-maximally entangled three-particle state. The average information transmitted in this scheme is explicitly given. We also generalize this scheme to the more general case.

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